

# REDUNDANT BLOW-UPS OF RATIONAL SURFACES WITH BIG ANTICANONICAL DIVISOR

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**ABSTRACT.** We completely classify redundant blow-ups appearing in the theory of rational surfaces with big anticanonical divisor due to Sakai. In particular, we construct a rational surface with big anticanonical divisor which is not a minimal resolution of a del Pezzo surface with only rational singularities, which gives a negative answer to a question raised in a paper by Testa, Várilly-Alvarado, and Velasco.

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## 1. INTRODUCTION

Throughout the paper, we work over an algebraically closed field  $k$  of arbitrary characteristic. Sakai ([S, Proposition 4.1 and Theorem 4.3]) proved that the anticanonical morphism  $f: S \rightarrow \tilde{S}$  of a big anticanonical rational surface, i.e., a smooth projective rational surface with big anticanonical divisor, factors through the minimal resolution of a del Pezzo surface with only rational singularities followed by a sequence of *redundant blow-ups*, which will be defined below. However, the existence of redundant points was not known before. In the present paper, we show the existence of redundant points (Theorem 1.2) by providing a systematic way of finding redundant points (Theorem 1.1).

Here, we briefly introduce the notion of redundant blow-up in general inspired by Sakai's work. Let  $S$  be a smooth projective surface. Assume that  $-K_S$  is pseudo-effective so that we have the Zariski decomposition  $-K_S = P + N$ . A point  $p$  in  $S$  is called a *redundant point* if  $\text{mult}_p N \geq 1$ . The blow-up  $f: \tilde{S} \rightarrow S$  at a redundant point  $p$  is called a *redundant blow-up*. For more detail, see Section 2.

To classify redundant blow-ups, we only need to know the information of the redundant points on the surface  $S$ . In many natural situations,  $S$  is a minimal

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resolution of a normal projective surface  $\bar{S}$  with nef anticanonical divisor. It turns out that the position of redundant points on  $S$  can be read off from the information of singularities on  $\bar{S}$ .

**Theorem 1.1.** *Let  $\bar{S}$  be a normal projective rational surface with nef anticanonical divisor, and let  $\tilde{S}$  be a surface obtained by a sequence of redundant blow-ups from the minimal resolution  $S$  of  $\bar{S}$ . Then, we have the following.*

- (1) *The number of surfaces obtained by a sequence of redundant blow-ups from  $S$  is finite if and only if  $\bar{S}$  contains at worst log terminal singularities.*
- (2) *If  $\bar{S}$  contains at worst log terminal singularities, then every redundant point on  $\tilde{S}$  lies on the intersection points of the two curves contracted by the morphism  $h: \tilde{S} \rightarrow \bar{S}$ .*
- (3) *If  $\bar{S}$  contains a non-log terminal singularity, then there is a curve in  $\tilde{S}$  contracted by the morphism  $h: \tilde{S} \rightarrow \bar{S}$  such that every point lying on this curve is a redundant point.*

Moreover, the existence of the redundant points can also be determined from the singularity types of  $\bar{S}$ .

**Theorem 1.2.** *Let  $\bar{S}$  be a normal projective rational surface with nef anticanonical divisor, and let  $g: S \rightarrow \bar{S}$  be its minimal resolution. Then,  $S$  has no redundant point if and only if  $\bar{S}$  contains at worst canonical singularities or log terminal singularities whose dual graphs are as follows:*

$$\begin{array}{ccccccc}
 \bullet & \bullet & \dots & \bullet & \bullet & \bullet & \bullet \\
 \hline
 -2 & -2 & & -2 & -3 & -2 & -2 & -3 & -2 & -2 & -3 & -2 & -2 & -4 & -n \\
 \hline
 & \underbrace{\hspace{1.5cm}}_{\alpha \ (\alpha \geq 1)} & & & & & & & & & & & & & (n \geq 3)
 \end{array}$$

It was shown that every big anticanonical rational surface has a finitely generated Cox ring in [TVAV, Theorem 1], [CS, Theorem 3], and the following question was raised in [TVAV].

**Question 1.3** ([TVAV, Remark 3]). *Is every big anticanonical rational surface the minimal resolution of a del Pezzo surface with only rational singularities?*

We give a negative answer to this question by explicitly constructing examples of redundant blow-ups (see Subsection 5.2).

**Theorem 1.4.** *For each  $n \geq 10$ , there exists a big anticanonical rational surface of Picard number  $n$  which is not a minimal resolution of a del Pezzo surface with only rational singularities.*

The remainder of this paper is organized as follows. We first define redundant blow-ups in Section 2. Then, Section 3 is devoted to the investigation of redundant points with respect to a given singularity, which leads us to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2 by calculating discrepancies. Finally, in Section 5, we construct examples of redundant blow-ups. In particular, we prove Theorem 1.4 in Subsection 5.2.

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## 2. BASIC SETUP

In this section, we define redundant blow-ups. Let  $S$  be a smooth projective surface, and let  $D$  be a  $\mathbb{Q}$ -divisor. The *Iitaka dimension* of  $D$  is given by

$$\kappa(D) := \max\{\dim \varphi|_{-nD}(S) : n \in \mathbb{N}\},$$

whose value is one of 2, 1, 0, and  $-\infty$ .

We will frequently use the notion of the *Zariski decomposition* of a pseudo-effective  $\mathbb{Q}$ -divisor  $D$  (see [S, Section 2] for details):  $D$  can be written uniquely as  $P + N$ , where  $P$  is a nef  $\mathbb{Q}$ -divisor,  $N$  is an effective  $\mathbb{Q}$ -divisor,  $P.N = 0$ , and the intersection matrix of the irreducible components of  $N$  is negative definite if  $N \neq 0$ .

Write an effective  $\mathbb{Q}$ -divisor  $D = \sum_{i=1}^n \alpha_i E_i$ , where  $E_i$  is a prime divisor for all  $1 \leq i \leq n$ . The *multiplicity* of  $D$  at a point  $p$  in  $S$  is defined by  $\text{mult}_p D := \sum_{i=1}^n \alpha_i \text{mult}_p E_i$ , where  $\text{mult}_p E_i$  denotes the usual multiplicity of  $E_i$  at  $p$ .

In the remainder of this subsection, we use the following notations. Let  $S$  be a smooth projective rational surface with  $\kappa(-K_S) \geq 0$ , and let  $-K_S = P + N$  be the Zariski decomposition. Let  $f: \tilde{S} \rightarrow S$  be a blow-up at a point  $p$  in  $S$  with the exceptional divisor  $E$ .

**Definition 2.1.** A point  $p$  is called *redundant* if  $\text{mult}_p N \geq 1$ . The blow-up  $f: \tilde{S} \rightarrow S$  at a redundant point  $p$  is called a *redundant blow-up*, and the exceptional curve  $E$  is called a *redundant curve*.

Note that we always have  $\kappa(-K_{\tilde{S}}) \leq \kappa(-K_S)$  in general. If  $f$  is a redundant blow-up, then  $\kappa(-K_{\tilde{S}}) \geq 0$  by [S, Lemma 6.9].

Now, we reformulate Sakai's result on redundant blow-ups. This will play a key role throughout the paper.

**Lemma 2.2** ([S, Corollary 6.7]). *Assume that  $\kappa(-K_{\tilde{S}}) \geq 0$  so that we have the Zariski decomposition  $-K_{\tilde{S}} = \tilde{P} + \tilde{N}$ . Then, the following are equivalent:*

- (1)  $f$  is a redundant blow-up.
- (2)  $\tilde{P} = f^*P$  and  $\tilde{N} = f^*N - E$ .

*Remark 2.3.* If  $-K_{\tilde{S}}$  is big, then  $\tilde{P}.E = 0$  if and only if  $f$  is a redundant blow-up by Lemma 2.2. Thus, our definition of redundant curves coincides with that of Sakai ([S, Definition 4.1]).

## 3. FINDING REDUNDANT POINTS

In this section, we focus on redundant points, and we prove Theorem 1.1 at the end. In what follows,  $\bar{S}$  denotes a normal projective rational surface such that  $-K_{\bar{S}}$  is nef, and  $g: S \rightarrow \bar{S}$  denotes the minimal resolution.

First, we recall some basic facts concerning normal singularities on surfaces. Let  $(\bar{S}, s)$  be a germ of a normal surface singularity, and let  $g: S \rightarrow \bar{S}$  be the minimal resolution. Denote the exceptional set by  $A = \pi^{-1}(s) = E_1 \cup \cdots \cup E_l$ , where each  $E_i$  is an irreducible component. Note that  $E_i^2 = -n_i \leq -1$ , where each  $n_i$  is an integer for  $i = 1, \dots, l$ . Now, we have

$$-K_S = g^*(-K_{\bar{S}}) + \sum_{i=1}^l a_i E_i,$$

where each  $a_i$  is a nonnegative rational number for all  $i = 1, \dots, l$ . Here, we call  $a_i$  the *discrepancy* of  $E_i$  with respect to  $\bar{S}$  for convenience in computation, even though  $-a_i$  is called the discrepancy in the literature. By the adjunction formula, this number can be obtained by the simultaneous linear equations:

$$\sum_{j=1}^l a_j E_j E_i = -K_S \cdot E_i = -n_i + 2 \text{ for } i = 1, \dots, l.$$

Thus, each discrepancy  $a_i$  can be calculated by the following matrix equation

$$\begin{pmatrix} -n_1 & E_2 E_1 & \cdots & E_l E_1 \\ E_2 E_1 & -n_2 & \cdots & E_l E_2 \\ \vdots & \vdots & \ddots & \vdots \\ E_l E_1 & E_l E_2 & \cdots & -n_l \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \end{pmatrix} = - \begin{pmatrix} n_1 - 2 \\ n_2 - 2 \\ \vdots \\ n_l - 2 \end{pmatrix}.$$

If an algebraic surface contains at worst rational singularities, then its singularities are isolated and the surface is  $\mathbb{Q}$ -factorial ([Bă, Theorem 4.6]), and the exceptional set  $A$  consists of smooth rational curves with a simple normal crossing (*snc* for short) support. Throughout this paper, we adopt the following standard terminologies.

**Definition 3.1.** (1) A singularity  $s$  is called *canonical* if  $a_i = 0$  for all  $i = 1, \dots, l$ .  
 (2) A singularity  $s$  is called *log terminal* if  $0 \leq a_i < 1$  for all  $i = 1, \dots, l$ .

We note that every log terminal singularity is rational ([KM, Theorem 4.12]). When  $\text{char}(k) = 0$ , a log terminal singularity is nothing but a quotient singularity ([KM, Proposition 4.18]).

We can get the Zariski decomposition of  $-K_S$  immediately by the assumption that  $-K_{\bar{S}}$  is nef.

**Lemma 3.2.** Let  $P = g^*(-K_{\bar{S}})$  and  $N = \sum_{i=1}^l a_i E_i$ , where each  $E_i$  is a  $g$ -exceptional curve and each  $a_i$  is the discrepancy of  $E_i$  with respect to  $\bar{S}$  for  $1 \leq i \leq l$ . Then,  $-K_S = P + N$  is the Zariski decomposition.

*Proof.* The pull-back of a nef divisor is again nef. Clearly  $P \cdot N = 0$  and the intersection matrix of irreducible components of  $N$  is negative definite.  $\square$

Suppose that  $S$  contains a redundant point  $p$ . Let  $f: \tilde{S} \rightarrow S$  be the redundant blow-up at  $p$  with the exceptional divisor  $E$ , and let  $-K_S = P + N$  and  $-K_{\tilde{S}} = \tilde{P} + \tilde{N}$  be the Zariski decompositions. Then, we obtain

$$\tilde{N} = \sum_{i=1}^l a_i \tilde{E}_i + (\text{mult}_p N - 1)E = \sum_{i=1}^l a_i \tilde{E}_i + \left( \sum_{p \in E_j} a_j - 1 \right) E,$$

where  $\tilde{E}_i$  is the strict transform of  $E_i$  for each  $1 \leq i \leq l$ . We note that  $\sum_{p \in E_j} a_j \geq 1$ , since  $\text{mult}_p N \geq 1$ .

**Remark 3.3.** If  $N$  has an *snc* support, then so does  $\tilde{N}$ . Thus, the negative part of the Zariski decomposition of the anticanonical divisor of a big anticanonical rational surface also has an *snc* support. This is no longer true for rational surfaces with anticanonical Iitaka dimension 1 or 0 (see Example 5.7).

In the remainder of this section, we will completely determine the location of redundant points on  $S$  in terms of the singularities of  $\bar{S}$ .

### 3.1. $\bar{S}$ contains at worst canonical singularities.

**Lemma 3.4.** *If  $\bar{S}$  has at worst canonical singularities, then  $S$  has no redundant points.*

*Proof.* By Definition 3.1,  $N = 0$ ; thus, the lemma follows.  $\square$

### 3.2. $\bar{S}$ contains at worst log terminal singularities.

**Lemma 3.5.** *If  $\bar{S}$  contains at worst log terminal singularities, then any surface  $\tilde{S}'$  obtained by a sequence of redundant blow-ups from  $S$  has finitely many redundant points, and every redundant point is an intersection point of two curves contracted by the morphism  $\tilde{S}' \rightarrow \bar{S}$ .*

*Proof.* By Definition 3.1,  $0 \leq a_i < 1$  for each  $1 \leq i \leq l$ . Finding a point  $p$  in  $S$  satisfying  $\text{mult}_p N \geq 1$  is equivalent to finding an intersection point of  $E_j$  and  $E_k$  such that  $a_j + a_k \geq 1$  for some  $1 \leq j \leq l$  and  $1 \leq k \leq l$ . Since the number of  $E_i$ 's is finite, the number of redundant points is also finite. In this case, we have

$$\tilde{N} = \sum_{i=1}^l a_i \tilde{E}_i + (a_j + a_k - 1)E.$$

Observe that  $a_i < 1$  for each  $1 \leq i \leq l$  and  $a_j + a_k - 1 < 1$ . Thus, after performing redundant blow-ups, the number of redundant points is still finite.  $\square$

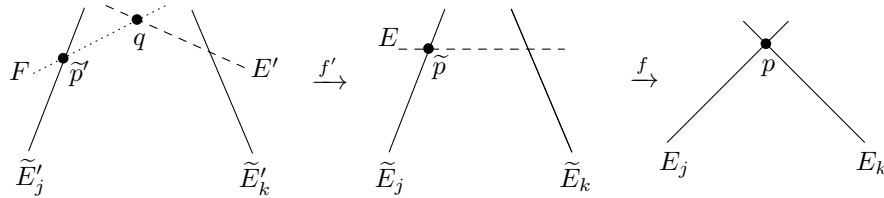
**Lemma 3.6.** *The number of surfaces obtained by a sequence of redundant blow-ups from  $S$  is finite.*

*Proof.* Let  $\tilde{p} := \tilde{E}_j \cap E$  the intersection point. Then, we have

$$\text{mult}_{\tilde{p}} \tilde{N} = a_j + (\text{mult}_p N - 1) = \text{mult}_p N - (1 - a_j) < \text{mult}_p N,$$

i.e., the multiplicity strictly decreases after redundant blow-ups.

Suppose that  $\text{mult}_{\tilde{p}} \tilde{N} \geq 1$ . Let  $f': \tilde{S}' \rightarrow \tilde{S}$  be a redundant blow-up at  $\tilde{p}$  with the redundant curve  $F$ , and let  $\tilde{E}'_j, \tilde{E}'_k$  and  $E'$  be the strict transforms of  $\tilde{E}_j, \tilde{E}_k$  and  $E$ , respectively. Let  $\tilde{p}' := \tilde{E}'_j \cap F$  and  $q := E' \cap F$  be the intersection points.



Let  $-K_{\tilde{S}'} = \tilde{P}' + \tilde{N}'$  be the Zariski decomposition. Then, we have

$$\begin{aligned} \text{mult}_{\tilde{p}} \tilde{N} - \text{mult}_{\tilde{p}'} \tilde{N}' &= \text{mult}_{\tilde{p}} \tilde{N} - \{\text{mult}_{\tilde{p}} \tilde{N} - (1 - a_j)\} = 1 - a_j, \\ \text{mult}_{\tilde{p}} \tilde{N} - \text{mult}_q \tilde{N}' &= \text{mult}_{\tilde{p}} \tilde{N} - \{(\text{mult}_{\tilde{p}} \tilde{N} - 1) + (\text{mult}_p N - 1)\} \\ &= 2 - \text{mult}_p N \\ &= 2 - a_j - a_k. \end{aligned}$$

Note that  $\text{mult}_p N - \text{mult}_{\tilde{p}} \tilde{N} = 1 - a_j < 2 - a_j - a_k$ , and hence, we obtain

$$\text{mult}_p N - \text{mult}_{\tilde{p}} \tilde{N} \leq \text{mult}_{\tilde{p}} \tilde{N} - \text{mult}_{\tilde{p}'} \tilde{N}',$$

and

$$\text{mult}_p N - \text{mult}_{\tilde{p}} \tilde{N} \leq \text{mult}_{\tilde{p}} \tilde{N} - \text{mult}_q \tilde{N}',$$

i.e., the difference of multiplicities increases after redundant blow-ups. Thus, the assertion immediately follows.  $\square$

There exist natural numbers  $M_j$  and  $M_k$  such that

$$\text{mult}_p N - M_j(1 - a_j) < 1 \text{ and } \text{mult}_p N - (M_j - 1)(1 - a_j) \geq 1,$$

and

$$\text{mult}_p N - M_k(1 - a_k) < 1 \text{ and } \text{mult}_p N - (M_k - 1)(1 - a_k) \geq 1.$$

Since  $\max\{M_j, M_k\}$  depends only on  $p$ , we denote it by  $M(p)$ .

**Corollary 3.7.** *The maximal length of sequences of redundant blow-ups from  $S$  is equal to*

$$\max_{p \in R} \{M(p)\},$$

where  $R$  is the set of all redundant points on  $S$ .

*Remark 3.8.* Corollary 3.7 shows that there is a bound of the length of sequences of redundant blow-ups for a given surface  $S$ . However, there is no global bound for  $M(p)$  (see Example 5.3; we have  $M_1 > m - 2 - \frac{2m}{n}$  and  $M_2 > n - 2 - \frac{2n}{m}$ , and hence,  $M(p)$  can be increased arbitrarily large as  $m$  and  $n$  goes to infinity).

### 3.3. $\tilde{S}$ contains worse than log terminal singularities.

**Lemma 3.9.** *If  $\tilde{S}$  contains a singularity that is not a log terminal singularity, then any surface  $\tilde{S}'$  obtained by a sequence of redundant blow-ups from  $S$  contains a curve  $C$  contracted by the morphism  $\tilde{S}' \rightarrow \tilde{S}$  such that every point in  $C$  is a redundant point. In particular,  $\tilde{S}'$  has infinitely many redundant points.*

*Proof.* By Definition 3.1, we can choose an integer  $k$  with  $1 \leq k \leq l$  such that  $a_k \geq 1$ , and hence, every point in  $E_k$  is a redundant point. After a redundant blow-up  $\tilde{S} \rightarrow S$  at a point  $\tilde{p}$  in  $E_k$ , every point in the proper transform of  $E_k$  is again a redundant point in  $\tilde{S}$ . In this way, the lemma easily follows.  $\square$

We are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemmas 3.5, 3.6, 3.9, the theorem holds.  $\square$

## 4. EXISTENCE OF REDUNDANT POINTS

In this section, we focus on the existence of redundant points, and we prove Theorem 1.2 at the end. As in Section 3, we use the following notations throughout this section:  $\tilde{S}$  denotes a normal projective rational surface such that  $-K_{\tilde{S}}$  is nef, and  $g: S \rightarrow \tilde{S}$  denotes the minimal resolution.

To prove Theorem 1.2, by Lemmas 3.4 and 3.9, we only need to consider the case when  $\tilde{S}$  has at worst log terminal singularities. Recall that in this case,  $p \in S$  is a redundant point if and only if there are two intersecting irreducible exceptional curves  $E_j$  and  $E_k$  in the minimal resolution such that the sum of discrepancies  $a_j + a_k \geq 1$ . Thus, it suffices to consider the problem locally near each singular point  $s$  in  $\tilde{S}$ , and hence, we can throughoutly assume that  $\tilde{S}$  contains only one log terminal singular point  $s$ .

In the case of characteristic zero, Brieskorn completely classified finite subgroups of  $GL(2, k)$  without quasi-reflections, i.e., he classified all the dual graphs of quotient singularities of surfaces ([Br]). It turns out that the complete list of the dual graphs of the log terminal surface singularities remains the same in arbitrary characteristic (see [A]). For the reader's convenience, we will give a detailed description of all possible types ( $A_{q,q_1}, D_{q,q_1}, T_m, O_m$  and  $I_m$ ) of dual graphs and discrepancies of log terminal singularities in the following subsections.

**4.1.  $A_{q,q_1}$ -type.** The dual graph of a log terminal singularity  $s \in \bar{S}$  of type  $A_{q,q_1}$  is

$$\begin{array}{ccccccc} E_1 & E_2 & E_3 & & E_{l-1} & E_l \\ \bullet & \bullet & \bullet & \dots & \bullet & \bullet \\ -n_1 & -n_2 & -n_3 & & -n_{l-1} & -n_l \end{array}$$

where each  $n_i \geq 2$  is an integer for all  $i$ . This singularity can be characterized by the so-called *Hirzebruch-Jung continued fraction*

$$\frac{q}{q_1} = [n_1, n_2, \dots, n_l] = n_1 - \frac{1}{n_2 - \frac{1}{\ddots - \frac{1}{n_l}}}.$$

The intersection matrix is

$$M(-n_1, \dots, -n_l) := \begin{pmatrix} -n_1 & 1 & 0 & \dots & \dots & 0 \\ 1 & -n_2 & 1 & \dots & \dots & 0 \\ 0 & 1 & -n_3 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -n_{l-1} & 1 \\ 0 & 0 & 0 & \dots & 1 & -n_l \end{pmatrix}.$$

For simplicity, we use the notation  $[n_1, \dots, n_l]$  to refer to a log terminal singularity of  $A_{q,q_1}$ -type. Note that  $[n_1, \dots, n_l]$  and  $[n_l, \dots, n_1]$  denote the same singularity.

We will use the following notation for convenience (cf. [HK2]).

- (1)  $q_{b_1, b_2, \dots, b_m} := |\det(M')|$  where  $M'$  is the  $(l-m) \times (l-m)$  matrix obtained by deleting  $-n_{b_1}, -n_{b_2}, \dots, -n_{b_m}$  from  $M(-n_1, \dots, -n_l)$ . For convenience, we also define  $q_{1, \dots, l} = |\det(M(\emptyset))| = 1$ .
- (2)  $u_s := q_{s, \dots, l} = |\det(M(-n_1, \dots, -n_{s-1}))|$  ( $2 \leq s \leq l$ ),  $u_0 = 0, u_1 = 1$ .
- (3)  $v_s := q_{1, \dots, s} = |\det(M(-n_{s+1}, \dots, -n_l))|$  ( $1 \leq s \leq l-1$ ),  $v_l = 1, v_{l+1} = 0$ .
- (4)  $q = |\det(M(-n_1, \dots, -n_l))| = u_{l+1} = v_0$ .
- (5)  $|[n_1, \dots, n_l]| := |\det(M(-n_1, \dots, -n_l))|$ .

It follows that  $q_1 = |\det(M(-n_2, \dots, -n_l))| = v_1$ . The following properties of Hirzebruch-Jung continued fractions will be used.

**Lemma 4.1.** *For  $1 \leq i \leq l$ , we have the following.*

- (1)  $a_i = 1 - \frac{u_i + v_i}{q}$ .
- (2)  $u_{i+1} = n_i u_i - u_{i-1}$ ,  $v_{i-1} = n_i v_i - v_{i+1}$ .
- (3)  $v_i u_{i+1} - v_{i+1} u_i = v_{i-1} u_i - v_i u_{i-1} = q$ .
- (4)  $|[n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_l]| = v_i u_i + |[n_1, n_2, \dots, n_l]| > q$ .
- (5) If  $l \geq 2$  and  $n_1 \geq 3$ , then  $v_1 + v_2 < q$ .

*Proof.* (1)-(4) are well-known facts (for (1), see [HK1, Lemma 2.2], and for (2)-(4), see [HK2, Lemma 2.4]). It is straightforward to check (5) as follows:

$$q = v_1 u_2 - v_2 u_1 = v_1 \cdot n_1 - v_2 = (n_1 - 1)v_1 + (v_1 - v_2) > 2v_1 > v_1 + v_2. \quad \square$$

Now, we give a criterion for checking  $a_k + a_{k+1} \geq 1$  for some  $k$ . Recall that the intersection point of  $E_K$  and  $E_{k+1}$  is a redundant point in  $S$ .

**Lemma 4.2.** *The following are equivalent.*

- (1)  $a_k + a_{k+1} \geq 1$  for some  $1 \leq k \leq l-1$ .
- (2)  $q \geq u_k + u_{k+1} + v_k + v_{k+1}$  for some  $1 \leq k \leq l-1$ .

*Proof.* By Lemma 4.1 (1), we have

$$a_k + a_{k+1} = \left(1 - \frac{u_k + v_k}{q}\right) + \left(1 - \frac{u_{k+1} + v_{k+1}}{q}\right) = 2 - \frac{u_k + u_{k+1} + v_k + v_{k+1}}{q},$$

and hence, the equivalence follows.  $\square$

The singularity  $\underbrace{[2, \dots, 2]}_l$  is a canonical singularity of type  $A_l$ , and hence,  $q = l+1$  and  $a_i = 0$  for all  $1 \leq i \leq l$ . On the other hand, the minimal resolution  $S$  of the singularity  $\underbrace{[2, \dots, 2, 3]}_{l-1}$  does not contain any redundant point. Indeed,  $q = 2l+1$  and  $u_i = i, v_i = 2l-2i+1$  for  $1 \leq i \leq l$  by using Lemma 4.1. Then, for  $1 \leq i \leq l-1$ ,

$$u_i + u_{i+1} + v_i + v_{i+1} = 4l - 2i + 1 > 2l + 1 = q,$$

and hence, by Lemma 4.2, there is no redundant point on  $S$ .

**Proposition 4.3.** *Let  $s \in \bar{S}$  be a log terminal singularity of type  $A_{q,q_1}$ . Then,  $S$  does not contain a redundant point if and only if  $s$  is of type  $\underbrace{[2, \dots, 2]}_\alpha (\alpha \geq 1)$ ,*

*$\underbrace{[2, \dots, 2, 3]}_\alpha (\alpha \geq 1)$ ,  $[2, 2, 3, 2]$ ,  $[2, 3, 2]$ ,  $[2, 4]$ , or  $[n]$  for  $n \geq 3$ .*

To prove the proposition, we need two more lemmas.

**Lemma 4.4.** *The following hold.*

- (1) *If  $s$  is the singularity  $\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\alpha} (\alpha \geq \beta \geq 1)$ , then  $q = \alpha\beta + 2\alpha + 2\beta + 3$  and the intersection point of  $E_\alpha$  and  $E_{\alpha+1}$  is a redundant point, except for  $[2, 2, 3, 2]$  and  $[2, 3, 2]$ .*
- (2) *If  $s$  is the singularity  $\underbrace{[2, \dots, 2, 3, 2, \dots, 2, 3, 2, \dots, 2]}_{\alpha} (\alpha \geq 0, \beta \geq 1 \text{ and } \gamma \geq 0)$ , then the intersection point of  $E_{\alpha+1}$  and  $E_{\alpha+2}$  is a redundant point.*
- (3) *If  $s$  is the singularity  $\underbrace{[2, \dots, 2, 4, 2, \dots, 2]}_{\alpha} (\alpha \geq \beta \geq 0)$ , then the intersection point of  $E_\alpha$  and  $E_{\alpha+1}$  is a redundant point, except for  $[2, 4]$  and  $[4]$ .*

*Proof.* The strategy is as follows. First, we compute  $u_\alpha, u_{\alpha+1}, u_{\alpha+2}, v_\alpha, v_{\alpha+1}, v_{\alpha+2}$  and  $q$  using Lemma 4.1. Second, we determine whether

$$q \geq u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1} \quad \text{or} \quad q \geq u_{\alpha+1} + u_{\alpha+2} + v_{\alpha+1} + v_{\alpha+2}$$

holds or not. Finally, by applying Lemma 4.2, we can find a redundant point.



(1) We have

$$\begin{aligned} u_\alpha &= |\underbrace{[2, \dots, 2]}_{\alpha-1}| = \alpha, v_\alpha = |\underbrace{[3, 2, \dots, 2]}_{\beta}| = 2\beta + 3 \\ u_{\alpha+1} &= |\underbrace{[2, \dots, 2]}_{\alpha}| = \alpha + 1, v_{\alpha+1} = |\underbrace{[2, \dots, 2]}_{\beta}| = \beta + 1. \end{aligned}$$

Then,  $u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1} = 2\alpha + 3\beta + 5$ . Using Lemma 4.1 (3), we obtain

$$q = v_\alpha u_{\alpha+1} - v_{\alpha+1} u_\alpha = (\alpha + 1)(2\beta + 3) - (\beta + 1)\alpha = \alpha\beta + 2\alpha + 2\beta + 3.$$

Then, we have  $q - (u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1}) = \alpha\beta - \beta - 2 = (\alpha - 1)\beta - 2$ . If  $\alpha\beta - \beta - 2 < 0$ , then  $(\alpha, \beta) = (1, 1)$  or  $(2, 1)$ . These cases are exactly  $[2, 2, 3, 2]$  and  $[2, 3, 2]$ , and we can easily check that there is no redundant point on  $S$  (see Table 1). Otherwise, we have  $q \geq (u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1})$ .

(2) We have

$$\begin{aligned} u_{\alpha+1} &= |\underbrace{[2, \dots, 2]}_{\alpha}| = \alpha + 1, v_{\alpha+1} = |\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\beta}| = \beta\gamma + 2\beta + 2\gamma + 3 \\ u_{\alpha+2} &= |\underbrace{[2, \dots, 2, 3]}_{\alpha}| = 2\alpha + 3, v_{\alpha+2} = |\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\beta-1}| = \beta\gamma + 2\beta + \gamma + 1. \end{aligned}$$

Then,  $u_{\alpha+1} + u_{\alpha+2} + v_{\alpha+1} + v_{\alpha+2} = 2\beta\gamma + 3\alpha + 4\beta + 3\gamma + 8$ . Using Lemma 4.1 (4), we obtain

$$\begin{aligned} q &= |\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\alpha+\beta+1}| + u_{\alpha+1}v_{\alpha+1} \\ &= (\alpha + \beta + 1)\gamma + 2(\alpha + \beta + 1) + 2\gamma + 3 + (\alpha + 1)(\beta\gamma + 2\beta + 2\gamma + 3) \\ &= \alpha\beta\gamma + 2\alpha\beta + 3\alpha\gamma + 2\beta\gamma + 5\alpha + 4\beta + 5\gamma + 8. \end{aligned}$$

It immediately follows that  $q \geq u_{\alpha+1} + u_{\alpha+2} + v_{\alpha+1} + v_{\alpha+2}$ .

(3) We have

$$\begin{aligned} u_\alpha &= |\underbrace{[2, \dots, 2]}_{\alpha-1}| = \alpha, v_\alpha = |\underbrace{[4, 2, \dots, 2]}_{\beta}| = 3\beta + 4 \\ u_{\alpha+1} &= |\underbrace{[2, \dots, 2]}_{\alpha}| = \alpha + 1, v_{\alpha+1} = |\underbrace{[2, \dots, 2]}_{\beta}| = \beta + 1. \end{aligned}$$

Then,  $u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1} = 2\alpha + 4\beta + 6$ . Using Lemma 4.1 (4), we obtain

$$q = |\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\alpha}| + u_{\alpha+1}v_{\alpha+1} = 2\alpha\beta + 3\alpha + 3\beta + 4.$$

Then, we have  $q - (u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1}) = 2\alpha\beta + \alpha - \beta - 2$ . If  $2\alpha\beta + \alpha - \beta - 2 < 0$ , then  $(\alpha, \beta) = (0, 0), (1, 0)$ . These cases are exactly  $[2, 4]$  and  $[4]$ , and we can easily check that there is no redundant point on  $S$  (see Table 1). Otherwise, we have  $q \geq (u_\alpha + u_{\alpha+1} + v_\alpha + v_{\alpha+1})$ .  $\square$

TABLE 1.

singularities	$[2, 2, 3, 2]$	$[2, 3, 2]$	$[2, 4]$	$[n]$
discrepancies	$(\frac{2}{11}, \frac{4}{11}, \frac{6}{11}, \frac{3}{11})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	$(\frac{2}{7}, \frac{4}{7})$	$(\frac{n-2}{n}) (n \geq 2)$

**Lemma 4.5.** *Suppose that for  $[n_1, \dots, n_j, \dots, n_k, \dots, n_l]$ , there are integers  $j$  and  $k$  with  $1 \leq j \leq k \leq l-1$  such that  $n_j \geq 3$  and  $u_j + u_{j+1} + v_j + v_{j+1} \leq q$ . Let*

$$[n'_1, \dots, n'_j, \dots, n'_{k-1}, n'_k, n'_{k+1}, \dots, n'_l] := [n_1, \dots, n_j, \dots, n_{k-1}, n_k+1, n_{k+1}, \dots, n_l].$$

*Then, we have  $u'_j + u'_{j+1} + v'_j + v'_{j+1} \leq q'$ .*

*Proof.* By Lemma 4.1,  $q' = q + u_k v_k$ . It suffices to show that  $u'_j + u'_{j+1} + v'_j + v'_{j+1} \leq q + u_k v_k$ . In order to calculate  $u'_j, u'_{j+1}, v'_j$ , and  $v'_{j+1}$ , we divide it into three cases.

Case 1:  $j+2 \leq k$ .

We have  $u'_j = u_j$  and  $u'_{j+1} = u_{j+1}$ . Moreover, we have

$$\begin{aligned} v'_j &= |[n_{j+1}, \dots, n_{k-1}, n_k+1, n_{k+1}, \dots, n_l]| = v_j + v_k |[n_{j+1}, \dots, n_{k-1}]| \text{ and} \\ v'_{j+1} &= |[n_{j+2}, \dots, n_{k-1}, n_k+1, n_{k+1}, \dots, n_l]| = v_{j+1} + v_k |[n_{j+2}, \dots, n_{k-1}]| \end{aligned}$$

Then,  $u'_j + u'_{j+1} + v'_j + v'_{j+1} \leq q'$  is equivalent to

$$\begin{aligned} u_j + u_{j+1} + v_j + v_{j+1} + v_k(|[n_{j+1}, \dots, n_{k-1}]| + |[n_{j+2}, \dots, n_{k-1}]|) \\ \leq q + u_k v_k = q + v_k |[n_j, \dots, n_{k-1}]|. \end{aligned}$$

The above inequality always holds by the assumption and Lemma 4.1 (5).

Case 2:  $j+1 = k$ .

We have  $u'_j = u_j$  and  $u'_{j+1} = u_{j+1}$ . Moreover, we have

$$v'_j = |[n_{j+1}+1, n_{j+2}, \dots, n_l]| = v_j + v_{j+1} \text{ and } v'_{j+1} = |[n_{j+2}, \dots, n_l]| = v_{j+1}.$$

Since  $v_{j+1} \leq u_{j+1} v_{j+1}$ , we obtain  $u'_j + u'_{j+1} + v'_j + v'_{j+1} \leq q + u_{j+1} v_{j+1}$ .

Case 3:  $j = k$ .

We have  $u'_j = u_j$  and  $u'_{j+1} = |[n_1, \dots, n_j+1]| = u_{j+1} + u_j$ . Moreover, we also have  $v'_j = v_j$  and  $v'_{j+1} = v_{j+1}$ . As in Case 2, we obtain  $u'_j + u'_{j+1} + v'_j + v'_{j+1} \leq q + u_{j+1} v_{j+1}$ .  $\square$

*Proof of Proposition 4.3.* First, we introduce notations for simplicity. Fix a natural number  $l$ . For integers  $n_1, \dots, n_l \geq 2$  and  $n'_1, \dots, n'_l \geq 2$ , we write  $[n_1, \dots, n_l] < [n'_1, \dots, n'_l]$  if and only if  $n_i \leq n'_i$  for all  $1 \leq i \leq l$  and  $n_j < n'_j$  for some  $1 \leq j \leq l$ .

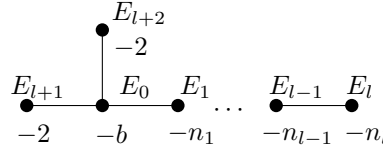
Suppose that  $s \in \bar{S}$  is the singularity  $[n_1, \dots, n_l]$  other than  $\underbrace{[2, \dots, 2]}_{\alpha}(\alpha \geq 1)$ ,

$\underbrace{[2, \dots, 2, 3]}_{\alpha}(\alpha \geq 1)$ ,  $[2, 2, 3, 2]$ ,  $[2, 3, 2]$ ,  $[2, 4]$ , or  $[n]$  for  $n \geq 3$ , in which cases  $S$  has no redundant point. Then,  $[n_1, \dots, n_l]$  is greater than or equal to  $\underbrace{[2, \dots, 2, 3, 2, \dots, 2]}_{\alpha}(\alpha \geq \beta \geq 1)$  (not equal to  $[2, 2, 3, 2]$  and  $[2, 3, 2]$ ),  $\underbrace{[2, \dots, 2, 3, 2, \dots, 2, 3, 2, \dots, 2]}_{\alpha}(\alpha \geq 0, \beta \geq 1 \text{ and } \gamma \geq 0)$ ,  $\underbrace{[2, \dots, 2, 4, 2, \dots, 2]}_{\alpha}(\alpha \geq \beta \geq 0)$  (not equal to  $[2, 4]$  and  $[4]$ ),  $[2, 2, 3, 3]$ ,  $[2, 3, 3, 2]$ ,  $[2, 3, 3]$ ,  $[3, 3]$ , or  $[2, 5]$ , in which cases  $S$  has a redundant point thanks to Lemma 4.4 and Table 2. Thus, by Lemma 4.5,  $[n_1, \dots, n_l]$  has a redundant point.  $\square$

TABLE 2.

singularities	[2, 2, 3, 3]	[2, 3, 3, 2]	[2, 3, 3]	[3, 3]	[2, 5]
discrepancies	$(\frac{2}{9}, \frac{4}{9}, \frac{6}{9}, \frac{5}{9})$	$(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})$	$(\frac{4}{13}, \frac{8}{13}, \frac{7}{13})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{3}, \frac{2}{3})$

4.2.  $D_{q,q_1}$ -**type**. The dual graph of a log terminal singularity  $s \in \bar{S}$  of  $D_{q,q_1}$ -type is



where  $\frac{q}{q_1} = [n_1, \dots, n_l]$ , and  $b \geq 2$  and  $n_i \leq 2$  are integers for all  $1 \leq i \leq l$ . The matrix equation for each discrepancy  $a_i$  is given by

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -b & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -n_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -n_2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix} \begin{pmatrix} a_{l+1} \\ a_{l+2} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_l \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ b-2 \\ n_1-2 \\ n_2-2 \\ n_3-2 \\ \vdots \\ n_l-2 \end{pmatrix}.$$

**Lemma 4.6** ([HK1, Lemma 3.7]). *We have the following.*

- (1)  $a_0 = 1 - \frac{1}{(b-1)q-q_1}$  and  $a_{l+1} = a_{l+2} = \frac{1}{2}a_0$ .
- (2)  $a_1 = 1 - \frac{b-1}{(b-1)q-q_1}$ .
- (3)  $a_l = 1 - \frac{(b-1)q_l-q_{1,l}}{(b-1)q-q_1}$  for  $l \geq 2$ .

**Proposition 4.7.** *Let  $s \in \bar{S}$  be a log terminal singularity of  $D_{q,q_1}$ -type. Then, we have the following four cases.*

- (1) *If  $b \geq 3$ , then  $a_0 + a_{l+1} \geq 1$ .*
- (2) *If  $b = 2$  and  $q \geq q_1 + 3$ , then  $a_0 + a_{l+1} \geq 1$ .*
- (3) *If  $b = 2$  and  $q = q_1 + 2$ , then  $a_0 = a_1 = \frac{1}{2}$ .*
- (4) *If  $b = 2$  and  $q = q_1 + 1$ , then  $s$  is a canonical singularity.*

*In particular, the minimal resolution  $S$  of  $\bar{S}$  always has a redundant point, unless  $s$  is a canonical singularity.*

*Proof.* Using Lemma 4.6, we get

$$a_0 + a_{l+1} = \frac{3}{2}a_0 = \frac{3}{2} \left( 1 - \frac{1}{(b-1)q-q_1} \right) \geq 1$$

which is equivalent to

$$(b-1)q - q_1 \geq 3.$$

If  $b \geq 3$ , then

$$(b-1)q - q_1 \geq 2q - q_1 > q \geq 2.$$

Thus,  $(b-1)q - q_1 \geq 3$ , that is,  $a_0 + a_{l+1} \geq 1$ , which proves (1).

Suppose that  $b = 2$ . If  $q - q_1 \geq 3$ , then it is still true that  $a_0 + a_{l+1} \geq 1$ . This proves (2).

Now, we assume that  $b = 2$  and  $q = q_1 + 2$ . The condition  $q = q_1 + 2$  is equivalent to  $(n_1 - 1)q_1 = q_{1,2} + 2$ . By simple calculation, we conclude that  $n_1 = 2$  and  $q_1 = q_{1,2} + 2$ . By induction,

$$[n_1, \dots, n_{l-1}, n_l] = [2, \dots, 2, 3].$$

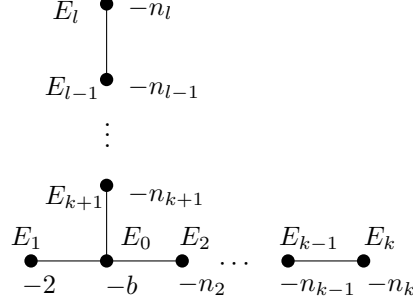
Using Lemma 4.6, we have  $a_0 = a_1 = \frac{1}{2}$ , which proves (3).

Finally, we assume that  $b = 2$  and  $q = q_1 + 1$ . As in the proof of (3), we obtain

$$[n_1, \dots, n_l] = [2, \dots, 2, 2].$$

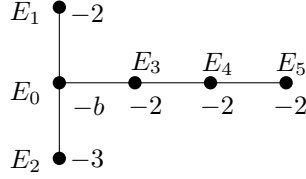
Thus,  $y$  is a canonical singularity of type  $D_{l+3}$ , which proves (4).  $\square$

**4.3.  $T_m$ ,  $O_m$ , and  $I_m$ -types.** We use the notation  $\langle b; q, q_1; q', q'_1 \rangle$  to refer to the following dual graph



where  $b \geq 2$  and  $n_i \geq 2$  are integers for all  $2 \leq i \leq l$ , and  $\frac{q}{q_1} = [n_2, \dots, n_k]$  and  $\frac{q'}{q'_1} = [n_{k+1}, \dots, n_l]$  are the Hirzebruch-Jung continued fractions. Each dual graph of a log terminal singularity  $s \in \bar{S}$  of  $T_m$ ,  $I_m$ , or  $O_m$ -types is one of three items from the top, eight items from the bottom, or the remaining items in Table 3, respectively.

For example, consider  $\langle b; 3, 1; 4, 3 \rangle$ , which is of  $O_m$ -type.



The matrix equation for discrepancies  $a_i$  is given by

$$\begin{pmatrix} -b & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 2-b \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to see that the solution  $(a_0, a_2, a_2, a_3, a_4, a_5)$  is

$$\left( \frac{12b-20}{12b-19}, \frac{6b-10}{12b-19}, \frac{8b-13}{12b-19}, \frac{9b-15}{12b-19}, \frac{6b-12}{12b-19}, \frac{3b-5}{12b-19} \right).$$

Since  $b \geq 2$ , we obtain  $a_0 + a_1 \geq 1$  by the following computation:

$$\frac{12b-20}{12b-19} + \frac{6b-10}{12b-19} = \frac{18b-30}{12b-19} \geq 1$$

is equivalent to

$$6b \geq 11.$$

Similarly, we calculate all discrepancies for 15 cases. For the reader's convenience, we list discrepancies  $(a_0, a_1, \dots, a_l)$  of all possible cases in Table 3. It is easy to check that  $a_0 + a_1 \geq 1$  for all cases provided that  $b \geq 2$  and  $y$  is not a canonical singularity. Thus, we obtain the following.

**Proposition 4.8.** *If  $s \in \bar{S}$  is a log terminal singularity of one of  $T_m$ ,  $O_m$ , and  $I_m$ -types, and  $s$  is not a canonical singularity, then  $a_0 + a_1 \geq 1$ .*

TABLE 3.

singularities	discrepancies $(a_0, a_1, \dots, a_l)$
$\langle b; 3, 1; 3, 1 \rangle$	$\left( \frac{6b-8}{6b-7}, \frac{3b-4}{6b-7}, \frac{4b-5}{6b-7}, \frac{4b-5}{6b-7} \right)$
$\langle b; 3, 1; 3, 2 \rangle$	$\left( \frac{6b-10}{6b-9}, \frac{3b-5}{6b-9}, \frac{12b-19}{18b-27}, \frac{12b-20}{18b-27}, \frac{6b-10}{18b-27} \right)$
$\langle b; 3, 2; 3, 2 \rangle$	$\left( \frac{6b-12}{6b-11}, \frac{3b-6}{6b-11}, \frac{4b-8}{6b-11}, \frac{2b-4}{6b-11}, \frac{4b-8}{6b-11}, \frac{2b-4}{6b-11} \right)$
$\langle b; 3, 2; 4, 3 \rangle$	$\left( \frac{12b-24}{12b-23}, \frac{6b-12}{12b-23}, \frac{8b-16}{12b-23}, \frac{4b-8}{12b-23}, \frac{9b-18}{12b-23}, \frac{6b-12}{12b-23}, \frac{3b-6}{12b-23} \right)$
$\langle b; 3, 1; 4, 3 \rangle$	$\left( \frac{12b-20}{12b-19}, \frac{6b-10}{12b-19}, \frac{8b-13}{12b-19}, \frac{9b-15}{12b-19}, \frac{6b-10}{12b-19}, \frac{3b-5}{12b-19} \right)$
$\langle b; 3, 2; 4, 1 \rangle$	$\left( \frac{12b-18}{12b-17}, \frac{6b-9}{12b-17}, \frac{8b-12}{12b-17}, \frac{4b-6}{12b-17}, \frac{9b-13}{12b-17} \right)$
$\langle b; 3, 1; 4, 1 \rangle$	$\left( \frac{12b-14}{12b-13}, \frac{6b-7}{12b-13}, \frac{8b-9}{12b-13}, \frac{9b-10}{12b-13} \right)$
$\langle b; 3, 2; 5, 4 \rangle$	$\left( \frac{30b-60}{30b-59}, \frac{15b-30}{30b-59}, \frac{20b-40}{30b-59}, \frac{10b-20}{30b-59}, \frac{24b-48}{30b-59}, \frac{18b-36}{30b-59}, \frac{12b-24}{30b-59}, \frac{6b-12}{30b-59} \right)$
$\langle b; 3, 2; 5, 3 \rangle$	$\left( \frac{30b-54}{30b-53}, \frac{15b-27}{30b-53}, \frac{20b-36}{30b-53}, \frac{10b-18}{30b-53}, \frac{24b-43}{30b-53}, \frac{18b-32}{30b-53} \right)$
$\langle b; 3, 1; 5, 4 \rangle$	$\left( \frac{30b-50}{30b-49}, \frac{15b-25}{30b-49}, \frac{20b-33}{30b-49}, \frac{24b-40}{30b-49}, \frac{18b-30}{30b-49}, \frac{12b-20}{30b-49}, \frac{6b-10}{30b-49} \right)$
$\langle b; 3, 2; 5, 2 \rangle$	$\left( \frac{30b-48}{30b-47}, \frac{15b-24}{30b-47}, \frac{20b-32}{30b-47}, \frac{10b-16}{30b-47}, \frac{24b-38}{30b-47}, \frac{12b-19}{30b-47} \right)$
$\langle b; 3, 1; 5, 3 \rangle$	$\left( \frac{30b-44}{30b-43}, \frac{15b-22}{30b-43}, \frac{20b-29}{30b-43}, \frac{24b-35}{30b-43}, \frac{18b-26}{30b-43} \right)$
$\langle b; 3, 2; 5, 1 \rangle$	$\left( \frac{30b-42}{30b-41}, \frac{15b-21}{30b-41}, \frac{20b-28}{30b-41}, \frac{10b-14}{30b-41}, \frac{24b-33}{30b-41} \right)$
$\langle b; 3, 1; 5, 2 \rangle$	$\left( \frac{30b-38}{30b-37}, \frac{15b-19}{30b-37}, \frac{20b-25}{30b-37}, \frac{24b-30}{30b-37}, \frac{12b-15}{30b-37} \right)$
$\langle b; 3, 1; 5, 1 \rangle$	$\left( \frac{30b-32}{30b-31}, \frac{15b-16}{30b-31}, \frac{20b-21}{30b-31}, \frac{24b-25}{30b-31} \right)$

We are ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* It is a consequence of Lemmas 3.9, 3.4, Propositions 4.3, 4.7, and 4.8.  $\square$

## 5. EXAMPLES OF REDUNDANT BLOW-UPS

In this section, we construct various examples of redundant blow-ups. In particular, we prove Theorem 1.4, i.e., we answer Question 1.3.

**5.1. Geometry of big anticanonical surfaces.** In this subsection, we briefly review some basic properties of big anticanonical surfaces. A smooth projective surface  $S$  is called a *big anticanonical surface* if the anticanonical divisor  $-K_S$  is big. Although not every anticanonical ring  $R(-K_X) := \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(-mK_S))$  of a big anticanonical surface  $S$  is finitely generated ([Bă, Lemma 14.39]), we can always define the anticanonical model of  $S$  using the Zariski decomposition  $-K_S = P + N$ . The morphism  $f: S \rightarrow \tilde{S}$  which contracts all curves  $C$  such that  $C \cdot P = 0$  is called the *anticanonical morphism*, and  $\tilde{S}$  is called the *anticanonical model* of  $S$ . Then,  $\tilde{S}$  is a normal projective surface ([Bă, 14.32]).

Note that the anticanonical ring  $R(-K_S)$  of a big anticanonical *rational* surface  $S$  is always finitely generated, and the anticanonical model  $\tilde{S} = \text{Proj } R(-K_S)$  is a del Pezzo surface, i.e.,  $-K_{\tilde{S}}$  is an ample  $\mathbb{Q}$ -Cartier divisor. In this case,  $\tilde{S}$  contains rational singularities by [S, Theorem 4.3]. Furthermore, we have the following.

**Proposition 5.1** ([S, Proposition 4.2]). *Let  $S$  be a big anticanonical rational surface. Then, there is a sequence of redundant blow-ups  $f: S \rightarrow S_0$ , where  $S_0$  is the minimal resolution of the anticanonical model of  $S$ .*

**5.2. Redundant blow-ups of big anticanonical rational surfaces.** In this subsection, we give an explicit construction of big anticanonical rational surfaces containing redundant points.

Before giving constructions, we explain how these examples give a negative answer to Question 1.3. Let  $S$  be a big anticanonical rational surface admitting a redundant blow-up  $f: \tilde{S} \rightarrow S$ . By Lemma 2.2, the anticanonical models  $\tilde{\tilde{S}}$  of  $\tilde{S}$  and  $S$  are the same. Suppose that  $\tilde{S}$  is a minimal resolution of a del Pezzo surface. Then, this minimal resolution is nothing but the anticanonical morphism to  $\tilde{S}$ . However, in view of Proposition 5.1,  $\tilde{S}$  cannot be the minimal resolution of  $\tilde{\tilde{S}}$  because the anticanonical morphism  $\tilde{\tilde{S}} \rightarrow \tilde{S}$  factors through  $S$ . Thus, the existence of redundant points on some big anticanonical rational surface answers to Question 1.3.

*Remark 5.2.* Let  $S$  be a big anticanonical rational surface, and let  $f: \tilde{S} \rightarrow S$  be a blow-up at  $p \in S$ . Even though  $f$  is not a redundant blow-up,  $\tilde{S}$  might be a big anticanonical rational surface. However, in this case, the anticanonical model of  $\tilde{S}$  is different from that of  $S$ .

In the following examples, we construct big anticanonical rational surfaces whose anticanonical models contain only log terminal singularities (Example 5.3) or contain at least one rational singularity that is not a log terminal singularity (Example 5.4).

**Example 5.3.** Let  $\pi: S(m, n) \rightarrow \mathbb{P}^2$  be the blow-up of  $\mathbb{P}^2$  at  $p_1^1, \dots, p_m^1$  on a line  $\bar{l}_1$  and  $p_1^2, \dots, p_n^2$  on a different line  $\bar{l}_2$  for  $m \geq n \geq 4$ . Assume that all the chosen points are distinct and away from the intersection point of  $\bar{l}_1$  and  $\bar{l}_2$ . Let  $l_i$  be the strict transform of  $\bar{l}_i$ ,  $E_j^i$  be the exceptional divisors of  $p_j^i$ , and  $l$  be the pull-back

of a general line in  $\mathbb{P}^2$ . Then, the anticanonical divisor of  $S(m, n)$  is given by

$$-K_{S(m, n)} = \pi^*(-K_{\mathbb{P}^2}) - \sum_{i, j} E_j^i = l + l_1 + l_2.$$

Let  $-K_{S(m, n)} = P + N$  be the Zariski decomposition. Then, we have

$$N = \frac{mn - m - 2n}{mn - m - n} l_1 + \frac{mn - 2m - n}{mn - m - n} l_2.$$

It is easy to see that  $S(m, n)$  is a big anticanonical rational surface. By contracting curves  $l_1$  and  $l_2$  on  $S(m, n)$ , we obtain the anticanonical model of  $S(m, n)$  which contains only log terminal singularities. Now, the intersection point  $p$  of  $l_1$  and  $l_2$  is a redundant point because

$$\text{mult}_p N = \frac{mn - m - 2n}{mn - m - n} + \frac{mn - 2m - n}{mn - m - n} \geq 1.$$

Thus, by blowing up at  $p$ , we obtain a redundant blow-up  $f: \tilde{S}(m, n) \rightarrow S(m, n)$ . The Picard number of  $\tilde{S}(m, n)$  is  $m + n + 2$ , and hence, it supports Theorem 1.4.

Singularities on log del Pezzo surfaces are classified in some cases (see e.g., [AN], [N], [K]). Thus, we can find redundant points on the minimal resolution of log del Pezzo surfaces.

**Example 5.4.** Let  $h: \mathbb{F}_n \rightarrow \mathbb{P}^1$  be the Hirzebruch surface with a section  $\bar{\sigma}$  of self-intersection  $-n \leq -2$ . Let  $k$  be an integer such that  $3 \leq k \leq n + 1$  and let  $a_1, \dots, a_k$  be positive integers such that  $\sum_{j=1}^k \frac{1}{a_j} < k - 2$ . Choose  $k$  distinct fibers  $\bar{F}_1, \dots, \bar{F}_k$  of  $h$ , and choose  $a_i$  distinct points  $p_1^i, \dots, p_{a_i}^i$  on  $\bar{F}_i \setminus \bar{\sigma}$ . Let  $S$  be the blow-up of  $\mathbb{F}_n$  at  $p_j^i$  where  $1 \leq i \leq k, 1 \leq j \leq a_i$ . Let  $\sigma$  be the strict transform of  $\bar{\sigma}$ , and let  $F_i$  be the strict transform of  $\bar{F}_i$ . Denote the pull-back of a general fiber of  $\mathbb{F}_n$  by  $F$ .

Then, we have the Zariski decomposition of the anticanonical divisor  $-K_S = P + N$  as follows:

$$P = \frac{n + 2 - k}{n - \sum \frac{1}{a_j}} \sigma + (n + 2 - k) F + \sum_{i=1}^k \frac{n + 2 - k}{a_i(n - \sum \frac{1}{a_j})} F_i,$$

$$N = \left( 2 - \frac{n + 2 - k}{n - \sum \frac{1}{a_j}} \right) \sigma + \sum_{i=1}^k \left( 1 - \frac{n + 2 - k}{a_i(n - \sum \frac{1}{a_j})} \right) F_i.$$

Thus,  $S$  is a big anticanonical rational surface, and every point  $p$  in  $\sigma$  is a redundant point because

$$\text{mult}_p N \geq 2 - \frac{n + 2 - k}{n - \sum \frac{1}{a_j}} > 1.$$

The construction of  $S$  appeared in Section 3 of [TVAV].

**5.3. Redundant blow-ups of other surfaces.** Throughout this subsection, we simply assume that  $k = \mathbb{C}$ . Here, we construct smooth projective surfaces with  $\kappa(-K) = 2, 1$ , or 0 containing redundant points.

**Example 5.5.** Let  $C$  be a smooth projective curve of genus  $g \geq 1$ , and let  $A$  be a divisor of degree  $e > 2g - 2$  on  $C$ . Consider the ruled surface  $X := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-A))$  with the ruling  $\pi: X \rightarrow C$ . Then,  $-K_X$  is big. It is easy to see that the negative part of the Zariski decomposition  $-K_X = P + N$  is given by  $N = (1 + \frac{2g-2}{e}) C_0$ ,

where  $C_0$  is a section of  $\pi$  corresponding to the canonical projection  $\mathcal{O}_C(-A) \oplus \mathcal{O}_C \rightarrow \mathcal{O}_C(-A)$ . Thus, every point on  $C_0$  is redundant.

**Example 5.6.** Let  $S$  be an extremal rational elliptic surface  $X_{321}$  in [MP, Theorem 4.1]. There is a singular fiber which consists of two smooth rational curves  $A$  and  $B$  meeting transversally at two points  $p$  and  $q$ . Let  $\pi: \tilde{S} \rightarrow S$  be the blow-up  $p$  with the exceptional divisor  $E$ . Since  $\text{mult}_p(A + B) = 2$ . By [AL, Lemma 4.4], we obtain the Zariski decomposition  $-K_{\tilde{S}} = P + N$ , where  $P = \frac{1}{2}\pi^*(-K_S)$  and  $N = \frac{1}{2}(\pi_*^{-1}A + \pi_*^{-1}B)$ . Then,  $\kappa(-K_{\tilde{S}}) = 1$ , and the intersection point  $q$  of  $\pi_*^{-1}A$  and  $\pi_*^{-1}B$  is redundant.

**Example 5.7.** Let  $S$  be an extremal rational elliptic surface  $X_{22}$  in [MP, Theorem 4.1]. The elliptic fibration has a singular fiber  $C$  which is a cuspidal rational curve and the unique section  $D$ . Let  $p$  be the intersection point of a singular fiber  $C$  and the section  $D$ , and let  $\pi: \tilde{S} \rightarrow S$  be the blow-up at  $p$  with the exceptional divisor  $E$ . Then, we obtain the Zariski decomposition  $-K_{\tilde{S}} = P + N$ , where  $P = 0$  and  $N = \pi_*^{-1}C$ . Thus,  $\kappa(-K_{\tilde{S}}) = 0$ , and every point on  $N$  is redundant.

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